DOI: 10.2478/auom-2018-0034
An. Şt. Univ. Ovidius Constanţa **Vol. 26(3),2018, 73–97**

Revisiting of some outstanding metric fixed point theorems via E-contraction

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Abstract

In this paper, we introduce the notion of α-ψ-contractive mapping of type E, to remedy of the weakness of the existing contraction mappings. We investigate the existence and uniqueness of a fixed point of such mappings. We also list some examples to illustrate our results that unify and generalize the several well-known results including the famous Banach contraction mapping principle.

1 Introduction and Preliminaries

Throughout the manuscript, we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the positive integers. Further, $\mathbb{R}$ represent the real numbers and $\mathbb{R}_0^+ := [0, \infty)$.

Let Ψ be the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Ψ_1) $\psi(t) < t$, for any $t > 0$.

(Ψ_2) φ is continuous at 0;

In the literatures, there are well-known examples for such functions, such as, comparison functions [3, 25] and (c)-comparison functions [3, 25].

Popescu [22] suggested the concept of α-orbital admissible as a refinement of the alpha-admissible notion, defined in [29, 11].

Key Words: α-ψ-contractive mapping of type , Fixed point, metric space.
2010 Mathematics Subject Classification: Primary 46T99, 47H10; Secondary 54H25.
Received: 21.11.2017
Accepted: 27.01.2018

Definition 1.1. *[22] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-orbital admissible if*

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1.$$

If the additional condition,

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1,$$

is fulfilled, then the α-admissible mapping T is called triangular α-orbital admissible.

Notice that each α-admissible mapping is an α-orbital admissible. For more details and counter examples, see e.g. [1, 4, 5, 11, 12, 16, 22].

In this manuscript, we define a new notion, α-ψ-contractive mapping of type E, and derive some existence and uniqueness fixed point theorems for such mappings. Our results remove some weakness of the existing fixed point theorems including the initial metric fixed point theorem, Banach contraction mapping principle. We list some immediate consequence and examples that illustrates our results.

2 Main results

Definition 2.1 (*cf.* [8])**.** *Let (X, d) be a metric space and $T : X \to X$ be given mapping. We say that T is an α-ψ-contractive mapping of type E if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(E(x, y)), \text{ for all } x, y \in X, \tag{1}$$

where

$$E(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)| \tag{2}$$

The following lemma is a standard argument to prove that the given sequence is Cauchy

Lemma 2.1. *(See e.g. [23]) Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n)$ is nonincreasing and $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$:*

$$d(x_{m_k}, x_{n_k}), d(x_{m_k+1}, x_{n_k+1}), d(x_{m_k-1}, x_{n_k}), d(x_{m_k}, x_{n_k-1}), d(x_{m_k-1}, x_{n_k-1})$$

Theorem 2.1. *Let (X, d) be a complete metric space. Let $T : X \to X$ be α-ψ-contractive mapping of type E satisfying the following conditions:*

(*i*) *T is triangular α-orbital admissible;*

(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(*iii*) *T is continuous.*

Then, there exists a fixed point x^* such that $Tx^* = x^*$.

Proof. By assumption (ii), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. We construct an iterative sequence $\{x_n\}$ such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Owing to the fact that T is $\alpha-$orbital admissible, we derive

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Recursively, we have

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{3}$$

Since T is triangular $\alpha-$orbital admissible, we find from (3) that

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ and } \alpha(x_{n+1}, x_{n+2}) \geq 1 \Rightarrow \alpha(x_n, x_{n+2}),$$

for any $n \in \mathbb{N}$. Inductively, we conclude that

$$\alpha(x_n, x_{n+j}) \geq 1, \text{ for all } n, j \in \mathbb{N}. \tag{4}$$

If $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}_0$, then $x^* = x_{n_0}$ forms a fixed point for T that the proof finishes. Hence, from now on, we assume that

$$x_n \neq x_{n+1} \text{ for all } n \in \mathbb{N}_0. \tag{5}$$

We shall prove that the sequence $\{d(x_n, x_{n+1})\}$ is monotone. By taking $x = x_n$ and $y = x_{n+1}$ in the inequality (1) and by regarding (3) and (5), we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n) d(Tx_{n-1}, Tx_n) \\ &\leq \psi(E(x_{n-1}, x_n)) \\ &= \psi(d(x_{n-1}, x_n) + |d(x_{n-1}, Tx_{n-1}) - d(x_n, Sx_n)|) \\ &\leq \psi(d(x_{n-1}, x_n) + |d(x_{n-1}, x_n) - d(x_n, x_{n+1})|) \end{aligned} \tag{6}$$

Suppose that $d(x_n, x_{n+1}) \geq d(x_{n-1}, x_n)$. In this case, the inequality (6), becomes

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq \psi(d(x_{n-1}, x_n) - d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\
&= \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}),
\end{aligned}
$$

a contradiction. Hence, we deduce that $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for each n. Moreover, we can derive an estimation from (6) that

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq \psi(E(x_{n-1}, x_n)) = \psi(2d(x_{n-1}, x_n) - d(x_n, x_{n+1})) \\
&\leq \psi(2d(x_{n-1}, x_n)) < 2d(x_{n-1}, x_n),
\end{aligned} \tag{7}
$$

for each n.

Since the sequence $\{d(x_n, x_{n+1})\}$ decreasing and bounded from below, we conclude that it converges to some nonnegative number $d \geq 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d.$$

Notice also that

$$\lim_{n\to\infty} E(x_n, x_{n+1}) = d.$$

We claim that $d = 0$. Suppose, on the contrary, that $d > 0$. Taking $\limsup$ of the inequality (7), and taking the basic condition (Ψ_1) into account, we get that

$$d \leq \psi(d) < d,$$

a contradiction. Hence, we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{8}$$

We shall proved that the sequence $\{x_n\}$ is Cauchy. Suppose, on the contrary, that there exist $\varepsilon > 0$ and sequences $\{x_n(k)\}$,$\{x_m(k)\}$ of positive integers such that $n(k) > m(k) > k$ and

$$\{d(x_{n(k)}, x_{m(k)}) \geq \varepsilon, d(x_{n(k)-1}, x_{m(k)}) < \varepsilon, \text{ for all } k \in \mathbb{N}.$$

Due to Lemma 2.1, we have

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon \tag{9}$$

On the other hand, from (4), we have $\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1$. Thus, from (1) and the assumption (ii), we deduce that

$$
\begin{aligned}
\varepsilon &\leq d(x_{n(k)}, x_{m(k)}) = d(Tx_{n(k)-1}, Tx_{m(k)-1}) \\
&\leq \alpha d(x_{n(k)-1}, x_{m(k)-1}) d(Tx_{n(k)-1}, Tx_{m(k)-1}) \\
&\leq \psi(E(x_{n(k)-1}, x_{m(k)-1}))
\end{aligned} \tag{10}
$$

Since

$$\begin{aligned}
E(x_{n(k)-1}, x_{m(k)-1}) &= d(x_{n(k)-1}, x_{m(k)-1}) + \\
&\qquad \left|d(x_{n(k)-1}, x_{Tn(k)-1}) - d(x_{m(k)-1}, x_{Tm(k)-1})\right| \\
&= d(x_{n(k)-1}, x_{m(k)-1}) + \\
&\qquad \left|d(x_{n(k)-1}, x_{n(k)}) - d(x_{m(k)-1}, x_{m(k)})\right|
\end{aligned}$$

using (9) and (8) we obtain that

$$\lim_{k\to\infty} E(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{11}$$

Finally, letting $k \to \infty$ in (10) and using (9) and (11), we get

$$\varepsilon \le \psi(\varepsilon). \tag{12}$$

On account of (Ψ_1), the inequality (12) turns into

$$\varepsilon \le \psi(\varepsilon) < \varepsilon,$$

which is a contradiction. Hence, we find that $\varepsilon = 0$. Therefore, x_n is a Cauchy sequences. By completeness of (X, d), the sequence x_n converges to some point $x^* \in X$ as $n \to \infty$. From the continuity of T, it follows that $x_{n+1} = Tx_n \to Tx^*)$ as $n \to \infty$. By the uniqueness of the limit, we get $x^* = Tx^*$, that is, x^* is a fixed point of T.

□

We say that a complete metric space (X, d) is regular if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $x_{n(k)}$ of x_n such that $\alpha(x_{n(k)}, x) \ge 1$ for all k.

As it is known well, the continuity condition is a very heavy condition. Like in [29], we realize that we can replace the continuity of the operator T by a regularity condition on a complete metric space (X, d).

Theorem 2.2. *Let (X, d) be a complete metric space. Let $T : X \to X$ be α-ψ-contractive mapping of type E satisfying the following conditions:*

(i) T is triangular α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;

(iii) (X, d) is regular.

Then, there exists a fixed point x^* such that $Tx^* = x^*$.

Proof. Following the proof of Theorem 2.1, we know that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By the hypothesis (iii) we deduce that there exists a subsequence $x_{n(k)}$ of x_n such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all k. Therefore, we have

$$\begin{aligned} d(x_{n(k)}, Tx^*) = d(Tx_{n(k)-1}, Tx^*) &\leq \alpha(x_{n(k)-1}, x^*) d(Tx_{n(k)-1}, Tx^*) \\ &\leq \psi(E(x_{n(k)-1}, x^*)) \end{aligned} \tag{13}$$

where

$$E(x_{n(k)-1}, x^*) = d(x_{n(k)-1}, x^*) + \left| d(x_{n(k)-1}, Tx_{n(k)-1}) - d(x^*, Tx^*) \right|$$

Now, we shall show that $Tx^* = x^*$. Suppose, on the contrary, that $Tx^* \neq x^*$ that is, $d(x^*, Tx^*) > 0$. Letting $k \to \infty$ in the above inequality, and taking (Ψ_1) into account, we find that

$$0 < d(x^*, Tx^*) \leq \psi(d(x^*, x^*)) < d(x^*, x^*),$$

which yields $d(x^*, Tx^*) = 0$, a contradiction. Therefore $Tx^* = x^*$. □

The assure the uniqueness of the fixed point, we will consider the following hypothesis

(U) for all $x \neq y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$, $\alpha(y, z) \geq 1$ and $\alpha(z, Tz) \geq 1$

Theorem 2.3. *The fixed point x^* of T, in Theorem 2.1 (resp. Theorem 2.2), is unique, if assume an additional condition (U).*

Proof. If we consider that $z = x_0$ we obtain that $\alpha(x_0, Tx_0) \geq 1$, so even hypotheses (ii), from Theorem 2.1 (resp. Theorem 2.2) so we obtain that x^* is a point fixed of T, where $x^* = lim_{n\to\infty} x_n = lim_{n\to\infty} T^n x$. Suppose that x^* and y^* are two fixed points of T such that $x^* = y^*$. Then, from (U) there exists $z \in X$ such that $\alpha(x^*, z) \geq 1$, $\alpha(y^*, z) \geq 1$ and $\alpha(z, Tz) \geq 1$. Since T is a triangular α-orbital admissible, we get that $\alpha(x^*, T^n z) \geq 1$ and $\alpha(y^*, T^n z) \geq 1$. Thus, from (1) we have

$$\begin{aligned} d(x^*, T^{n+1} z) = d(Tx^*, T(T^n z)) &\leq \alpha(x^*, T^n z) d(Tx^*, T(T^n z)) \\ &\leq \psi(E(x^*, T^n z)) < E(x^*, T^n z) \end{aligned} \tag{14}$$

This imply that

$$d(x^*, T^{n+1}z) < d(x^*, T^n z) + \left|d(x^*, Tx^*) - d(T^n x, T^{n+1}x)\right|$$

By Theorem 2.2 we deduce that the sequence $T^n z$ converges to a fixed point z^* of T. Letting $n \to \infty$ in the above inequality, we get $d(x_*, z^*) < d(x_*, z^*)$. This implies that $d(x^*, z^*) = 0$ so $x^* = z^*$. Similarly, we get $y^* = z^*$. Hence, $x^* = y^*$, which is a contradiction. □

2.1 Examples

Now, we shall consider some examples that illustrate and support our main results.

Example 2.1. *Let $X = \mathbb{R}$. Consider the self mapping $T : X \to X$ such that*

$$Tx = \begin{cases} \frac{3x}{8} & \text{if } x \in [0,1], \\ \frac{x^2+13}{16} & \text{otherwise} \end{cases}$$

Define a metric $d : X \times X \to \mathbb{R}_0^+$ as $d(x, y) = \frac{|x-y|}{2}$
Define $\alpha : X \times X \to \mathbb{R}_0^+$ such that

$$\alpha(x, y) = \begin{cases} 4 & \text{if } (x, y) \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the self-mapping T is continuous.

Further, several well-known contraction types does not hold for $(x, y) \in [0, 1]$, that is,

$$d(Tx, Ty) = \frac{3|x-y|}{16} \text{ and hence } \alpha(x, y)d(Tx, Ty) = \frac{3|x-y|}{4},$$

but $d(x, y) = \frac{|x-y|}{2}$. Hence, there is no $k \in [0, 1)$ or $\psi \in \Psi$ such that the conditions of α-ψ-contractive mapping or Banach contraction mapping principle are fulfilled.

On the other hand, for all $x, y \in [0, 1]$, one can easily derive that

$$E(x, y) = \frac{|x-y|}{2} + \frac{1}{2}\left||x - \frac{3x}{8}| - |y - \frac{3y}{8}|\right| = \frac{13|x-y|}{16}, \text{ for all } x, y \in [0, 1],$$

which yields that $T : X \to X$ is an α-ψ-contractive mapping of type E with $\psi(t) = at$ with $a \geq \frac{12}{13}$, for example $a = \frac{25}{26}$. Indeed, we have for $x, y \in [0, 1]$

$$\alpha(x, y)d(Tx, Ty) = \frac{3|x-y|}{4} \leq \frac{25|x-y|}{32} = \frac{25E(x, y)}{26} = \psi(E(x, y)).$$

Note that if $x, y \in \mathbb{R} \setminus [0,1]$*, then the result is provided easily from the fact that* $\alpha(x,y) = 0$.

Let us check that T *is* α*-orbital admissible:*

$$\alpha(x,Tx) = \alpha(x, \tfrac{x}{4}) \geq 1 \quad \Rightarrow \alpha(Tx, T^2x) = \alpha(\tfrac{x}{4}, \tfrac{x}{16}) \geq 1.$$

As a second step, let us prove that T *is triangular* α*-orbital admissible:*

$$\alpha(x,y) \geq 1 \text{ and } \alpha(y,Ty) = \alpha(y, \tfrac{y}{4}) \geq 1 \quad \Rightarrow \alpha(x,Ty) = \alpha(x, \tfrac{y}{4}) \geq 1,$$

Thus, the first condition (i) *of Theorem 2.1 is satisfied. The second condition* (ii) *of Theorem 2.1 is also fulfilled. Indeed, for* $x_0 = 0$*, we have* $\alpha(0, T0) = \alpha(0,0) = 5 \geq 1$.

Thus, all conditions of Theorem 2.1 are satisfied. Here, $T0 = 0$ *is the fixed point of* T.

Example 2.2. *Let* $X = A \cup \mathbb{R}_0^+$ *where* $A = \{a,b,c\}$*. Consider the self mapping* $T : X \to X$ *such that*

$$Tx = \begin{cases} x+1 & \text{if } x \in \mathbb{R}_0^+, \\ c & \text{if } x = a, \\ b & \text{if } x \in \{b,c\}. \end{cases}$$

Define a metric $d : X \times X \to \mathbb{R}_0^+$ *as*

$$d(x,y) = \begin{cases} |x-y| & \text{if } x,y \in \mathbb{R}_0^+, \\ 1 & \text{if } (x,y) \in \{(a,b),(b,a)\} \\ 3 & \text{if } (x,y) \in \{(a,c),(c,a)\}, \\ 2 & \text{if } (x,y) \in \{(b,c),(c,b)\}, \\ 0 & \text{if } (x,y) \in \{(a,a),(b,b),(c,c)\}, \\ 0 & \text{if } (x,y) \in A \times \mathbb{R}_0^+ \cup \mathbb{R}_0^+ \times A. \end{cases}$$

Define $\alpha : X \times X \to \mathbb{R}_0^+$ *such that*

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \{(a,c),(a,b),(b,b),(c,b)\} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to get that

$$E(a,b) = 1 + |3-0| = 4, E(a,c) = 3 + |3-2| = 4, E(b,c) = 2 + |0-2| = 4.$$

From the above calculations, we can easily conclude that $T : X \to X$ *is an* α*-*ψ*-contractive mapping of type* E*, for* $\psi(t) = \frac{t}{2}$*. Indeed, we have the following possibilities:*

$$\begin{aligned} \alpha(a,b)d(Ta,Tb) = 2 &\leq \tfrac{E(a,b)}{2} = 2, \\ \alpha(a,c)d(Ta,Tc) = 2 &\leq \tfrac{E(a,c)}{2} = \tfrac{4}{2} = 2, \\ \alpha(b,c)d(Tb,Tc) = 0 &\leq \tfrac{E(b,c)}{2} = \tfrac{4}{2} = 2. \end{aligned}$$

Thus, the condition (1) is satisfied for all $x, y \in A$. Notice that the condition (1) is fulfilled trivially for $x, y \in \mathbb{R}_0^+$, since $\alpha(x,y) = 0$ for $x, y \in \mathbb{R}_0^+$.

Let us check that T is α-orbital admissible:

$$\begin{aligned} \alpha(a,Ta) = \alpha(a,c) \geq 1 \quad &\Rightarrow \alpha(Ta,T^2a) = \alpha(c,b) \geq 1 \\ \alpha(b,Tb) = \alpha(b,b) \geq 1 \quad &\Rightarrow \alpha(Tb,T^2b) = \alpha(b,b) \geq 1 \\ \alpha(c,Tc) = \alpha(c,b) \geq 1 \quad &\Rightarrow \alpha(Tc,T^2c) = \alpha(b,b) \geq 1 \end{aligned}$$

As a second step, let us prove that T is triangular α-orbital admissible:

$$\begin{aligned} \alpha(a,b) \geq 1 \text{ and } \alpha(b,Tb) = \alpha(b,b) \geq 1 \quad &\Rightarrow \alpha(a,Tb) = \alpha(a,b) \geq 1, \\ \alpha(a,c) \geq 1 \text{ and } \alpha(c,Tc) = \alpha(c,b) \geq 1 \quad &\Rightarrow \alpha(a,Tc) = \alpha(a,b) \geq 1, \\ \alpha(b,b) \geq 1 \text{ and } \alpha(b,Tb) = \alpha(b,b) \geq 1 \quad &\Rightarrow \alpha(b,Tb) = \alpha(b,b) \geq 1, \\ \alpha(c,b) \geq 1 \text{ and } \alpha(b,Tb) = \alpha(b,b) \geq 1 \quad &\Rightarrow \alpha(c,Tb) = \alpha(b,b) \geq 1. \end{aligned}$$

Thus, the first condition (i) of Theorem 2.2 is satisfied. The second condition (ii) of Theorem 2.2 is also fulfilled. Indeed, for any $x_0 \in A$, we have $\alpha(x_0, Tx_0) \geq 1$.

It is also easy to see that (X,d) is regular. Indeed, whatever the initial point $x_0 \in A$ is chosen, the sequence $\{x_n\}$ tends to b. By definition of the auxiliary function α, we have

$$\alpha(a,b) \geq 1, \alpha(b,b) \geq 1 \text{ and } \alpha(c,b) \geq 1.$$

Thus, all conditions of Theorem 2.2 are provided. Notice that $Tb = b$ is the fixed point of T.

Example 2.3. *Let $X = \left[0, \frac{2}{3}\right] \cup [1,4]$ equipped with a metric $d : X \times X \to \mathbb{R}_0^+$ such that $d(x,y) = \frac{|x-y|}{2}$. Consider the self mapping $T : X \to X$ such that*

$$Tx = \begin{cases} \frac{x}{3} & \text{if } x \in \left[0, \frac{2}{3}\right], \\ \\ \frac{5}{2} - \frac{x}{4} & \text{if } x \in [1,4]. \end{cases}$$

We define $\alpha : X \times X \to \mathbb{R}_0^+$ such that

$$\alpha(x,y) = \begin{cases} 4 & \text{if } x, y \in \left[0, \frac{2}{3}\right] \\ 1 & \text{if } x, y \in [1,3] \\ 2 & \text{if } (x,y) \in \{(0,4),(1,4)\}, \\ 0 & \text{otherwise.} \end{cases}$$

First of all, we remark that, for $x, y \in \left[0, \frac{2}{3}\right]$,

$$d(x,y) = \frac{|x-y|}{2} = \frac{3|x-y|}{6}, \text{ and } d(Tx,Ty) = \frac{1}{2}\left|\frac{x}{3} - \frac{y}{3}\right| = \frac{|x-y|}{6}$$

Thus, we have

$$\alpha(x,y)d(Tx,Ty) = \frac{4|x-y|}{6}$$

Hence, there is no $\psi \in \Psi$ *to provide that* T *is* $\alpha - \psi$ *contraction.*

We have consider the following two cases:

(i) *for all* $x, y \in \left[0, \frac{2}{3}\right]$

$$E(x,y) = \frac{|x-y|}{2} + \frac{1}{2}\left||x - \frac{x}{3}| - |y - \frac{y}{3}|\right| = \frac{5\,|x-y|}{6}.$$

Consequently, we conclude that T *is* $\alpha - \psi$ *contraction type of* E*, with* $\psi(t) = at$*, where* $a > \frac{4}{5}$*. In particular, for the choice of* $\psi(t) = \frac{9}{10}t$*, we have*

$$\alpha(x,y)d(Tx,Ty) = \frac{4\,|x-y|}{6} \leq \psi(\frac{9}{12}\,|x-y|) = \frac{9}{10}E(x,y) = \psi(E(x,y)).$$

which yields that $T : X \to X$ *is an* α-ψ-*contractive mapping of type* E *with* $\psi(t) = \frac{9}{10}t$*.*

(ii) *for* $x, y \in [1, 3]$*, we have*

$$d(Tx,Ty) = \frac{1}{2}\left|\frac{x}{4} - \frac{y}{4}\right| = \frac{|x-y|}{8} \text{ and } d(x,y) = \frac{|x-y|}{2}$$

and $\alpha(x,y)d(Tx,Ty) = \frac{|x-y|}{8}$ *Moreover, we find that*

$$E(x,y) = \frac{|x-y|}{2} + \frac{1}{2}\left||x - \frac{x}{4}| - |y - \frac{y}{4}|\right| = \frac{7\,|x-y|}{8}.$$

Hence, we conclude that

$$\alpha(x,y)d(Tx,Ty) = \frac{|x-y|}{8} \leq \frac{9}{10}\left(\frac{7\,|x-y|}{8}\right) = \frac{9}{10}E(x,y) = \psi(E(x,y)).$$

(iii) *for* $x = 0$ *and* $y = 4$*, we have*

$$d(T0,T4) = \frac{1}{2}\left|0 - \frac{3}{2}\right| = \frac{3}{4} \text{ and } d(0,4) = \frac{|0-4|}{2} = 2 \text{ and } \alpha(0,4)d(T0,T4) = \frac{6}{4}.$$

Moreover, we find that

$$E(0,4) = \frac{|0-4|}{2} + \frac{1}{2}\left||0-0| - |4 - \frac{3}{2}|\right| = 2 + \frac{5}{4} = \frac{13}{4}.$$

Hence, we deduce that

$$\alpha(0,4)d(T0,T4) = \frac{6}{4} \leq \frac{117}{40} = \frac{9}{10}E(0,4) = \psi(E(0,4)).$$

(iv) for $x = 1$ and $y = 4$, we have

$$d(T1,T4) = \frac{1}{2}\left|\frac{9}{4} - \frac{3}{2}\right| = \frac{3}{8} \text{ and } d(1,4) = \frac{|1-4|}{2} = \frac{3}{4}$$

and $\alpha(1,4)d(T1,T4) = \frac{6}{8}$. Moreover, we find that

$$E(1,4) = \frac{|1-4|}{2} + \frac{1}{2}\left||1-\frac{9}{4}| - |4-\frac{3}{2}|\right| = \frac{3}{2} + \frac{5}{8} = \frac{17}{8}.$$

Hence, we deduce that

$$\alpha(1,4)d(T1,T4) = \frac{6}{8} \leq \frac{153}{80} = \frac{9}{10}E(1,4) = \psi(E(1,4)).$$

Notice that for any other possibilities, the result is provided easily from the fact that $\alpha(x,y) = 0$.

Let us check that T is α-orbital admissible:

(i) if $x \in \left[0, \frac{2}{3}\right]$

$$\alpha(x,Tx) = \alpha(x, \tfrac{x}{3}) \geq 1 \quad \Rightarrow \alpha(Tx, T^2x) = \alpha(\tfrac{x}{3}, \tfrac{x}{9}) \geq 1.$$

(ii) if $x \in [1,3]$ then $Tx \in [1,3]$ and

$$\alpha(x,Tx) = \geq 1 \quad \Rightarrow \alpha(Tx, T^2x) \geq 1.$$

As a second step, let us prove that T is triangular α-orbital admissible:

(i) if $x \in \left[0, \frac{2}{3}\right]$

$$\alpha(x,y) \geq 1 \text{ and } \alpha(y,Ty) = \alpha(y, \tfrac{y}{3}) \geq 1 \quad \Rightarrow \alpha(x,Ty) = \alpha(x, \tfrac{y}{3}) \geq 1,$$

(ii) if $x, y \in [1,3]$ then $Ty \in [1,3]$ and

$$\alpha(x,y) \geq 1 \text{ and } \alpha(y,Ty) = \alpha(y, \tfrac{5}{2} - \tfrac{y}{4}) \geq 1 \quad \Rightarrow \alpha(x,Ty) = \alpha(x, \tfrac{5}{2} - \tfrac{y}{4}) \geq 1,$$

Thus, the first condition (i) of Theorem 2.2 is satisfied. The second condition (ii) of Theorem 2.2 is also fulfilled. Indeed, for $x_0 = 0$, we have $\alpha(0,T0) = \alpha(0,0) = 1 \geq 1$.

It is also easy to see that (X,d) is regular. Indeed let x_n be a sequence in X such that for all n and $x_n \to x$ as $n \to \infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all n, by the definition of α, we have $x_n \in [0,1]$ for all n and $x \in [0,1]$. Then,

$$\alpha(x_n, x) = 1.$$

Thus, all conditions of Theorem 2.2 are provided and so we derive that T has a fixed point.

We shall show that the condition (U) are not satisfied. Thus, we could not guarantee the uniqueness of a fixed point.

$$\alpha(0,4) \geq 1, \alpha(1,4) \geq 1 \quad \text{and } \alpha(4,T4)\alpha(4,\tfrac{3}{2}) = 0,$$

Notice that $T0 = 0$ and $T2 = 2$ are the fixed points of T.

Example 2.4. *Let $X = \mathbb{R}$ equipped with a metric $d : X \times X \to \mathbb{R}_0^+$ such that $d(x,y) = \frac{|x-y|}{2}$. Consider the self mapping $T : X \to X$ such that*

$$Tx = \begin{cases} \frac{x}{4} & \text{if } x \in [0,1], \\ 0 & \text{otherwise} \end{cases}$$

We define $\alpha : X \times X \to \mathbb{R}_0^+$ such that

$$\alpha(x,y) = \begin{cases} 5 & \text{if } (x,y) \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the self-mapping T is not continuous at $x = 1$.

Further, several well-known contraction types does not hold for $(x,y) \in [0,1]$, that is,

$$d(Tx,Ty) = \frac{|x-y|}{8} \text{ and hence } \alpha(x,y)d(Tx,Ty) = \frac{5\,|x-y|}{8}$$

but $d(x,y) = \frac{|x-y|}{2}$. Hence, there is no $\psi \in \Psi$ such that the conditions of α-ψ-contractive mapping is fulfilled.

On the other hand, for all $x,y \in [0,1]$, one can easily derive that

$$E(x,y) = \frac{|x-y|}{2} + \frac{1}{2}\left||x - \frac{x}{4}| - |y - \frac{y}{4}|\right| = \frac{7\,|x-y|}{8}, \text{ for all } x,y \in [0,1],$$

which yields that $T : X \to X$ is an α-ψ-contractive mapping of type E with $\psi(t) = at$ with $a \geq \frac{5}{7}$, for example $a = \frac{13}{14}$. Indeed, we have for $x,y \in [0,1]$

$$\alpha(x,y)d(Tx,Ty) = \frac{5\,|x-y|}{8} \leq \frac{13\,|x-y|}{16} = \frac{13E(x,y)}{14} = \psi(E(x,y)).$$

Note that if $x,y \in \mathbb{R} \setminus [0,1]$, then the result is provided easily from the fact that $\alpha(x,y) = 0$.

Let us check that T is α-orbital admissible:

$$\alpha(x,Tx) = \alpha(x,\tfrac{x}{4}) \geq 1 \quad \Rightarrow \alpha(Tx,T^2x) = \alpha(\tfrac{x}{4},\tfrac{x}{16}) \geq 1.$$

As a second step, let us prove that T is triangular α-orbital admissible:

$$\alpha(a,y) \geq 1 \text{ and } \alpha(y,Ty) = \alpha(y, \tfrac{y}{4}) \geq 1 \quad \Rightarrow \alpha(x,Ty) = \alpha(x, \tfrac{y}{4}) \geq 1,$$

Thus, the first condition (i) of Theorem 2.2 is satisfied. The second condition (ii) of Theorem 2.2 is also fulfilled. Indeed, for $x_0 = 0$, we have $\alpha(0,T0) = \alpha(0,0) = 5 \geq 1$.

Finally, it is also easy to see that (X,d) is regular. Indeed let x_n be a sequence in X such that for all n and $x_n \to x$ as $n \to \infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all n, by the definition of α, we have $x_n \in [0,1]$ for all n and $x \in [0,1]$. Then,

$$\alpha(x_n, x) = 5 \geq 1.$$

Thus, all conditions of Theorem 2.2 are provided. Notice that $T0 = 0$ is the fixed point of T.

Moreover, because for any $z \in [0,1]$, $Tz = \frac{z}{4} \in [0,1]$ we deduce that for all $x, y \in [0,1]$, $x \neq y$ there exist $z \in [0,1]$ such that $\alpha(x,z) \geq 1, \alpha(y,z) \geq 1$ and $\alpha(z,Tz) \geq 1$, so condition (U) is satisfied. Thus, Theorem 2.3 guarantee the uniqueness of the fixed point.

3 Outcomes of the main results

In this section, we shall list some basic consequences of the main results.

3.1 Outcomes of the main results in the setting of standard metric structure.

Theorem 3.1. *Let (X,d) be a complete metric space and $\psi \in \Psi$. Suppose that a continuous self-mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \leq \psi(E(x,y)), \text{ for all } x,y \in X, \tag{15}$$

where

$$E(x,y) = d(x,y) + |d(x,Tx) - d(y,Ty)|.$$

Then, there exists a fixed point x^* such that $Tx^* = x^*$.

Proof. It is sufficient to take $\alpha(x,y) = 1$ for all $x, y \in X$ in Theorem 2.3. □

Theorem 3.2. *Let (X,d) be a complete metric space and $k \in [0,1)$. Suppose that a continuous self-mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \leq kE(x,y), \text{ for all } x,y \in X, \tag{16}$$

where

$$E(x,y) = d(x,y) + |d(x,Tx) - d(y,Ty)| \,.$$

Then, there exists a fixed point x^* such that $Tx^* = x^*$.

Proof. It is sufficient to take $\psi(t) = kt$ in Theorem 3.1. □

In the following theorems, the continuity condition of the self-mapping is not necessary, since the contraction condition (17) and (18) imply the continuity of the mentioned self-mapping.

Theorem 3.3. *Let (X,d) be a complete metric space and $\psi \in \Psi$. Suppose that a self-mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \le \psi(d(x,y)), \text{ for all } x,y \in X. \tag{17}$$

Then, there exists a fixed point x^* such that $Tx^* = x^*$.

Proof. It follows from Theorem 3.1, by regarding the monotonicity of the function ψ, that is,

$$d(Tx,Ty) \le \psi(d(x,y)) \le \psi(E(x,y)), \text{ for all } x,y \in X.$$

□

Theorem 3.4. *Let (X,d) be a complete metric space and $k \in [0,1)$. Suppose that a self-mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \le kd(x,y), \text{ for all } x,y \in X. \tag{18}$$

Then, there exists a fixed point x^* such that $Tx^* = x^*$.

Proof. It follows from Theorem 3.3, by letting $\psi(t) = kt$. □

3.2 Outcomes of the main results in the setting of metric spaces endowed with a partial order

In the last decades, one of exciting research topics for the metric fixed point theory researchers is to investigate the existence and uniqueness of a fixed point of certain mapping in context of metric spaces endowed with partial orders, see e.g.[24, 18] We shall show that Theorem 2.3 infer various existing fixed point results on a metric space endowed with a partial order. For this purpose, we, first, recollect some basic concepts.

Definition 3.1. *For a partially ordered non-empty set $(X,\preceq)$, the self-mapping $T : X \to X$ T is called nondecreasing with respect to $\preceq$ if*

$$x,y \in X, \; x \preceq y \Longrightarrow Tx \preceq Ty.$$

Definition 3.2. *A sequence $\{x_n\}$ in a partially ordered set $(X, \preceq)$ is called nondecreasing with respect to $\preceq$, if $x_n \preceq x_{n+1}$ for all n.*

Definition 3.3. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, \preceq, d)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.*

Suppose that $(X, \preceq)$ is a partially ordered set and d be a metric on X. We say that $(X, \preceq)$ have a property of (S) if it fulfills the following condition

(S) for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$,

For the simplicity, we shall use the notation $(X, d, \preceq)$ to represent the partially ordered set $(X, \preceq)$ equipped with a metric d. The triple $(X, d, \preceq)$ is called metric spaces endowed with a partial order.

Theorem 3.5. *Let $(X, d, \preceq)$ be a metric spaces endowed with a partial order, where (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(E(x, y), \tag{19}$$

for all $x, y \in X$ with $x \succeq y$, where

$$E(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)|\,.$$

Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

Then, T has a fixed point. Moreover, if $(X, \preceq)$ have a property of (S), the observed fixed point is unique.

Proof. Consider the mapping $\alpha : X \times X \to [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x \preceq y \text{ or } x \succeq y, \\ 0 \text{ otherwise.} \end{cases}$$

It is obvious that T $T : X \to X$ is an α-ψ-contractive mapping of type E, that is,

$$\alpha(x, y)d(Tx, Ty) \leq \psi(E_T(x, y)),$$

for all $x, y \in X$. From condition (i), the definition of α yields that $\alpha(x_0, Tx_0) \geq 1$.

Moreover, for all $x, y \in X$, from the monotone property of T, we have

$$\alpha(x,y) \geq 1 \Longrightarrow x \succeq y \text{ or } x \preceq y \Longrightarrow Tx \succeq Ty \text{ or } Tx \preceq Ty \Longrightarrow \alpha(Tx,Ty) \geq 1.$$

Consequently, T is $\alpha-$orbital admissible.

For a last step, we examine the following cases: If T is continuous, the existence of a fixed point follows from Theorem 2.1. Suppose now that $(X, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. Due to regularity, there is a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. Hence, we have $\alpha(x_{n(k)}, x) \geq 1$ for all k. So, the existence of a fixed point follows from Theorem 2.2.

For the uniqueness, let $x, y \in X$. By assumption of the theorem, there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, which yields that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$. Consequently, we conclude the uniqueness of the fixed point by Theorem 2.3. □

Theorem 3.6. *Let $(X, d, \preceq)$ be a metric spaces endowed with a partial order, where (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kE(x,y),$$

for all $x, y \in X$ with $x \succeq y$, where

$$E(x,y) = d(x,y) + |d(x,Tx) - d(y,Ty)|.$$

Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T is continuous or $(X, \preceq, d)$ is regular.

Then, T has a fixed point. Moreover, if $(X, \preceq)$ have a property of (S), the observed fixed point is unique.

Theorem 3.7. *Let $(X, d, \preceq)$ be a metric spaces endowed with a partial order, where (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \psi(d(x,y)),$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) T *is continuous or* $(X, \preceq, d)$ *is regular.*

Then, T *has a fixed point. Moreover, if* $(X, \preceq)$ *have a property of* (S)*, the observed fixed point is unique.*

Theorem 3.8. *Let* $(X, d, \preceq)$ *be a metric spaces endowed with a partial order, where* (X, d) *is complete. Let* $T : X \to X$ *be a nondecreasing mapping with respect to* $\preceq$*. Suppose that there exists a* $k \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq kd(x, y),$$

for all $x, y \in X$ *with* $x \succeq y$*. Suppose also that the following conditions hold:*

(i) there exists $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*;*

(ii) T *is continuous or* $(X, \preceq, d)$ *is regular.*

Then, T *has a fixed point. Moreover, if* $(X, \preceq)$ *have a property of* (S)*, the observed fixed point is unique.*

3.3 Outcomes of the main results in the setting of the cyclic contractive mappings

Investigation of the existence and uniqueness of a fixed point of certain cyclic contractive mappings was initiated by Kirk, Srinivasan and Veeramani [17]. Following this pioneer paper [17], this trend has been appreciated by a number of authors (see e.g. [26, 13, 14] and the related references therein).

Here, we shall indicate that our main result, Theorem 2.3, infer a fixed point theorems for cyclic contractive mappings.

Theorem 3.9. *Suppose that* $\{A_i\}_{i=1}^2$ *are nonempty closed subsets of a complete metric space* (X, d) *and* $T : Y \to Y$ *is a given mapping with* $Y = A_1 \cup A_2$*. If the the following conditions are fulfilled*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(II) there exists a function $\psi \in \Psi$ *such that*

$$d(Tx, Ty) \leq \psi(E(x, y)), \text{ for all } (x, y) \in A_1 \times A_2,$$

where $E(x, y)$ *is defined as in (2),*

then, T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

Proof. The proof consists of several steps. Step 1. the pair (Y, d) forms a complete metric space since A_1 and A_2 are closed subsets of (X, d).

Step 2. We shall indicate that T is an $\alpha - \psi$ contractive mapping type of E. For this purpose, we specify the mapping $\alpha : Y \times Y \to [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 1 \text{ if } (x, y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 \text{ otherwise.} \end{cases}$$

Regarding (II) and α, we are able to write

$$\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y)),$$

for all $x, y \in Y$. In other words, T is an $\alpha - \psi$ contractive mapping type of E.

Step 3. We shall show that T is $\alpha-$admissible. Suppose that $(x, y) \in Y \times Y$ with $\alpha(x, y) \geq 1$. For the case $(x, y) \in A_1 \times A_2$, from (I), $(Tx, Ty) \in A_2 \times A_1$, which yields that $\alpha(Tx, Ty) \geq 1$. For the other case, $(x, y) \in A_2 \times A_1$, again from (I), $(Tx, Ty) \in A_1 \times A_2$, which implies that $\alpha(Tx, Ty) \geq 1$. So, we find that $\alpha(Tx, Ty) \geq 1$ whenever $\alpha(x, y) \geq 1$.

Step 4. We notice that for any $a \in A_1$, from (I), we get $(a, Ta) \in A_1 \times A_2$, and hence $\alpha(a, Ta) \geq 1$.

Step 5. We shall show that (X, d) is regular. Suppose that $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. On account of the definition of the α mapping, we find

$$(x_n, x_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \text{ for all } n.$$

Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set with respect to the Euclidean metric, we derive that

$$(x, x) \in (A_1 \times A_2) \cup (A_2 \times A_1),$$

which yields that $x \in A_1 \cap A_2$. Consequently, we have $\alpha(x_n, x) \geq 1$ for all n.

Finally, assume that $x, y \in \text{Fix}(T)$. From (I), we find that $x, y \in A_1 \cap A_2$. As a result, we conclude that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, for any $z \in Y$. Thus, condition (U) is satisfied.

Thus, all the hypotheses of Theorem 2.3 are fulfilled that guarantees the existence and uniqueness of a fixed point of T in $A_1 \cap A_2$ (from (I)). □

As an immediate outcome of Theorem 3.9 is the following:

Theorem 3.10. *Suppose that $\{A_i\}_{i=1}^2$ are nonempty closed subsets of a complete metric space (X, d) and $T : Y \to Y$ is a given mapping with $Y = A_1 \cup A_2$. If the the following conditions are fulfilled*

(I) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;

(II) there exists a function $\psi \in \Psi$ *such that*

$$d(Tx, Ty) \leq \psi(d(x, y)), \text{ for all } (x, y) \in A_1 \times A_2,$$

where $d(x, y)$ *is defined as in (2),*

then, T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

Theorem 3.11. *Suppose that* $\{A_i\}_{i=1}^2$ *are nonempty closed subsets of a complete metric space* (X, d) *and* $T : Y \to Y$ *is a given mapping with* $Y = A_1 \cup A_2$*. If the the following conditions are fulfilled*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(II) there exists a $k \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq kE(x, y), \text{ for all } (x, y) \in A_1 \times A_2,$$

where $E(x, y)$ *is defined as in (2),*

then, T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

The following is the main result of [17].

Theorem 3.12. *Suppose that* $\{A_i\}_{i=1}^2$ *are nonempty closed subsets of a complete metric space* (X, d) *and* $T : Y \to Y$ *is a given mapping with* $Y = A_1 \cup A_2$*. If the the following conditions are fulfilled*

(I) $T(A_1) \subseteq A_2$ *and* $T(A_2) \subseteq A_1$*;*

(II) there exists a $k \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } (x, y) \in A_1 \times A_2,$$

then, T *has a unique fixed point that belongs to* $A_1 \cap A_2$*.*

4 An Applications to the solutions of periodic boundary value problems of first order

In this section we examine the existence and uniqueness of solutions of periodic boundary value problems of first order. Although these problems have been investigated under different conditions in [19], [20]-[2], we suggest weaker condition for the existence and uniqueness conditions.

We consider $X = C[0,T]$ with the partial ordering

$$\begin{aligned} & x \preceq y \Rightarrow x(t) \leq y(t) \quad \text{for all} \quad t \in [0,T] \\ & d(x,y) = \sup\{|x(t) - y(t)|, t \in [0,T]\}. \end{aligned} \tag{20}$$

equipped with the metric

$$d(x,y) = \sup\{|x(t) - y(t)|, t \in [0,T]\}. \tag{21}$$

The space $(X, d, \preceq)$ satisfies the condition (S). Indeed, it is obvious that for every pair $x(t), y(t)$ in X, we have $x(t) \preceq \max\{x(t), y(t)\}$ and $y(t) \preceq \max\{x(t), y(t)\}$.

We shall discuss the following first order periodic boundary value problem

$$\begin{cases} x'(t) &= f(t, x(t)), \quad t \in [0,T] \\ x(0) &= x(T). \end{cases} \tag{22}$$

Definition 4.1. *A lower solution of the problem* (22) *is a function* $x(t) \in C[0,T]$ *fulfilling*

$$\begin{cases} x'(t) &\leq f(t, x(t)), \quad t \in [0,T] \\ x(0) &\leq x(T). \end{cases} \tag{23}$$

An upper solution the problem (22) *is a function* $x(t) \in C[0,T] \times \mathbb{R}$ *satisfying*

$$\begin{cases} x'(t) &\geq f(t, x(t)), \quad t \in [0,T] \\ x(0) &\geq x(T). \end{cases} \tag{24}$$

Note that the problem (22) can be written as

$$\begin{cases} x'(t) + \lambda x(t) &= f(t, x(t)) + \lambda x(t), \quad t \in [0,T] \\ x(0) &= x(T). \end{cases} \tag{25}$$

This problem is equivalent to the integral equation

$$x(t) = \int_0^T G(t,s)[f(s, x(s) + \lambda x(s)]ds, \tag{26}$$

where $G(t,s)$ is the Green function defined by

$$G(t,s) = \begin{cases} \dfrac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}, & 0 \leq s < t \leq T \\ \dfrac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}, & 0 \leq t < s \leq T. \end{cases} \tag{27}$$

Now, we state a theorem for the existence and uniqueness of a solution of the problem (23).

Theorem 4.1. *We take the periodic boundary value problem* (22) *into account. Suppose that f is continuous and that there exists $\lambda > 0$ such that for all $x, y \in C[0,T]$ satisfying $x \leq y$, the following condition holds:*

$$0 \leq f(t, y(t)) + \lambda y(t) - f(t, x(t)) - \lambda x(t) \leq \psi((y - x)), \tag{28}$$

for some $\psi \in \Psi$, $\lambda \in [0, \infty)$, such that $0 < \psi(u) < u < \lambda$, for all $u \in \mathbb{R}^+$. If the problem (22) *has a lower solution, then it has a unique solution.*

Proof. We set the map $F : C[0,T] \to \mathbb{R}$ as follows

$$Fx(t) = \int_0^T G(t,s)[f(s, x(s) + \lambda x(s)]ds, \tag{29}$$

where $G(t,s)$ is the Green function given in (27). Then the solution of the problem (22) is the fixed point of F. Suppose that $x \leq y$ are functions in $C[0,T] \times \mathbb{R}$ satisfying (28). Furthermore, since f fulfills (28), we get

$$\begin{aligned} Fx(t) &= \int_0^T G(t,s)[f(s, x(s)) + \lambda x(s)]ds \\ &\leq \int_0^T G(t,s)[f(s, y(s)) + \lambda y(s)]ds = Fy(t), \end{aligned} \tag{30}$$

that is, F is nondecreasing. Let us examine

$$\begin{aligned} d(Fy, Fx) &= \sup \left| \int_0^T G(t,s)[f(s, y(s)) + \lambda y(s) - f(s, x(s)) - \lambda x(s)]ds \right| \\ &\leq \sup \textstyle\int_0^T G(t,s)\psi(|y(s) - x(s)|)ds \\ &\leq \psi(d(x,y)) \textstyle\int_0^T G(t,s)ds = \dfrac{\psi(d(x,y))}{\lambda} \leq \dfrac{\psi(E(x,y))}{\lambda} \end{aligned}$$

where $E(x,y) = d(x,y) + |d(x,Tx) - d(y,Ty)|$. By selecting λ in a way that $0 < \psi(E(x,y)) < \lambda$ we conclude that the nondecreasing map F fulfills the condition (19) of Corollary 3.5. As a next step, we shall indicate that $x_0 \leq Fx_0$ for some $x_0 \in X$. Since the problem (22) has a lower solution, then, there exists $x_0 \in X$ provides (23). Hence, we get that

$$\begin{aligned} x_0'(t) + \lambda x_0(t) &\leq f(t, x_0(t)) + \lambda x_0(t), \quad t \in [0,T] \\ x_(0) &\leq x_0(T). \end{aligned} \tag{31}$$

Multiplying both sides by $e^{\lambda t}$ and then integrating from 0 to t we derive

$$x_0(t)e^{\lambda t} \leq x_0(0) + \int_0^t e^{\lambda s}[x_0(s) + f(s, x_0(s))]ds. \tag{32}$$

By using the inequality $x_0(0) \leq x_0(T)$ we get

$$x_0(0)e^{\lambda T} \leq x_0(T)e^{\lambda T} \leq x_0(0) + \int_0^T e^{\lambda s}[x_0(s) + f(s, x_0(s))]ds,$$

or equivalently,

$$x_0(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}[x_0(s) + f(s, x_0(s))]ds. \tag{33}$$

Combining (32) and (33) we get

$$\begin{aligned} x_0(t) \quad &\leq \quad \int_0^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}[x_0(s) + f(s, x_0(s))]ds \\ &+ \quad \int_0^t e^{\lambda(s-t)}[x_0(s) + f(s, x_0(s))]ds \\ &= \quad \int_0^T G(t,s)[x_0(s) + f(s, x_0(s))]ds \end{aligned} \tag{34}$$

where $G(s,t)$ is the Green's function given in (28). So, we have

$$x_0(t) \leq Fx_0(t)$$

for the lower solution $x_0(t)$ of (22). Then, by the Corollary 3.5 the map F has a unique fixed point. Accordingly, the boundary value problem (22) has a unique solution. □

Example 4.1. *Let $X = C[0,1]$ with a partial order $x \preceq y$ and a metric*

$$d(x,y) = \sup\{|x(t) - y(t)|, t \in [0,1]\}.$$

Consider the BVP

$$\begin{cases} x'(t) &= -x^2e^t + 1, \quad t \in [0,1] \\ x(0) &= x(1). \end{cases}$$

For this specific example the function

$$f(t,x) = -x^2e^t + x + 1$$

fulfils the condition

$$0 \leq f(t, y(t)) + \lambda y(t) - f(t, x(t)) - \lambda x(t) \leq k(y(t) - x(t))$$

for all $0 < x \leq y$*. Indeed,*

$$\begin{aligned} f(t, y(t)) + \lambda y(t) - f(t, x(t)) - \lambda x(t) &= -y^2 e^t + 1 + \lambda y - (-x^2 e^t + 1) - \lambda x \\ &= [\lambda - (y + x)](y - x) \\ &\leq [\lambda - M](y - x) \\ &\leq \psi(y - x) \end{aligned}$$

for $\psi(t) = (\lambda - M)t$*, where* $M = \min_{t \in [0,1]} \{(x + y)e^t\}$*. Observe that* $x_0(t) = 0$ *is a lower solution of the BVP. Clearly,*

$$x_0'(t) = 0 \leq -x^2 e^t + 1, t \in [0, 1],$$

and

$$x_0(0) = 0 = x_0(1).$$

By the Theorem 4.1, the BVP has a unique solution.

References

[1] H.Aydi, M. Jellali, E. Karapınar, Common fixed points for generalized α-implicit contractions in partial metric spaces: Consequences and application, RACSAM - Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matemiticas, September 2015, Volume 109, Issue 2, pp 367-384

[2] A. Amini-Harandi and H. Emami, *A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations*, Nonlinear Anal., **72**, 2238–2242, (2010).

[3] V. Berinde, Contracţii generalizate şi aplicaţii, Editura Club Press 22, Baia Mare, 1997.

[4] N. Bilgili, E. Karapınar, *A note on "common fixed points for (ψ, α, β)-weakly contractive mappings in generalized metric spaces"*, Fixed Point Theory Appl., **2013** (2013), Article ID 287.

[5] SH. Cho, JS. Bae, E. Karapinar, Fixed point theorems for α-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl. 2013, Article ID 329 (2013).

[6] B.S. Choudhury, N. Metiya, Fixed point and common fixed point results in ordered cone metric spaces, Analele Universitatii "Ovidius" Constanta-Seria Matematica, Volume 20, Issue 1, Pages 5572, ISSN (Online) 1844-0835, DOI: https://doi.org/10.2478/v10309-012-0005-8, 2012.

[7] J. Cringanu, D. Paşca, A fixed point method for a class of nonlinear equations involving a duality mapping defined on product spaces, Electronic Journal of Differential Equations, ISSN 1072-6691, 2013.

[8] A. Fulga and A. Proca, A new Generalization of Wardowski Fixed Point Theorem in Complete Metric Spaces, Advances in the Theory of Nonlinear Analysis and its Applications, Volume 1, Issue 1, Pages: 57-63, Year: 2017, Article Id: 2017:5 PDF

[9] T. Gnana Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65**, 1379–1393, (2006).

[10] J. Harjani and K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal., **71**, 3403–3410, (2009).

[11] E. Karapınar, P. Kuman, P. Salimi, On $\alpha - \psi$-Meri-Keeler contractive mappings, Fixed Point Theory Appl. (2013), 2013:94.

[12] E. Karapınar and B. Samet. Generalized $(\alpha - \psi)$-Contractive Type Mappings and Related Fixed Point Theorems with Applications Abstract and Applied Analysis Volume 2012 (2012), Article ID 793486, 17 pages

[13] E. Karapınar, Fixed point theory for cyclic weak ϕ-contraction, Appl. Math. Lett. 24 (6) (2011) 822–825.

[14] E. Karapınar, K. Sadaranagni, Fixed point theory for cyclic $(\phi$-$\psi)$-contractions, Fixed Point Theory Appl. 2011, 2011:69.

[15] E. Karapinar, I. M, Erhan and U. Aksoy Weak -contractions on partially ordered metric spaces and applications to boundary value problems Boundary Value Problems, 2014, 2014:149 https://doi.org/10.1186/s13661-014-0149-8

[16] E. Karapınar, H.H. Alsulami and M. Noorwali, Some extensions for Geraghty type contractive mappings Journal of Inequalities and Applications 2015, 2015:303 (26 September 2015)

[17] W.A. Kirk, P.S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory. 4(1) (2003) 79–89.

[18] J.J. Nieto, R. Rodríguez-López, Contractive Mapping Theorems in Partially Ordered Sets and Applications to Ordinary Differential Equations. Order. 22 (2005) 223–239.

[19] J. J. Nieto and R. R. Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22**, 223–239, (2005).

[20] J. J. Nieto, R. L. Pouso, and R. Rodrguez-Lpez, *Fixed point theorems in ordered abstract spaces*, Proc. of the American Math. Soc., **135**, 2505–2517, (2007).

[21] T. P. Petru, A. Petruşel and J.-C. Yao, Ulam-Hyers stability for operatorial equations and inclusions via nonself operators, Taiwanese Journal of Mathematics, Vol. **15**, No. 5, pp. 2195-2212, October 2011.

[22] O. Popescu, Some new fixed point theorems for α-Geraghty contractive type maps in metric spaces, Fixed Point Theory Appl. 2014, 2014:190

[23] S. Radenović, Z. Kadelburg, D. Jandrlić and A. Jandrlić, Some results on weak contraction maps, Bulletin of the Iranian Mathematical Society Vol. 38 No. 3 (2012), pp 625-645.

[24] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2003) 1435–1443.

[25] I. A. Rus, Generalized contractions and applications, Cluj University Press, Cluj-Napoca, 2001.

[26] I.A. Rus, Cyclic representations and fixed points, Ann. T. Popoviciu, Seminar Funct. Eq. Approx. Convexity 3(2005) 171–178.

[27] I. A. Rus, The theory of a metrical fixed point theorem: theoretical and applicative relevances, Fixed Point Theory, 9(2008), No. 2, 541-559.

[28] I. A. Rus, Remarks on Ulam stability of the operatorial equations, Fixed Point Theory, 10(2009), No. 2, 305-320.

[29] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Analysis 75 (2012), 2154-2165.

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